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Asymptotic Analysis of Autoresonant Oscillator Chains

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Abstract

This paper investigates the emergence of autoresonance (AR) with growing energy in a chain of time-invariant linear oscillators weakly coupled to a nonlinear actuator (the Duffing oscillator) driven by an external periodic force. Two types of forcing are studied: (1) harmonic forcing with constant frequency is applied to an actuator with slowly-varying parameters; (2) harmonic forcing with a slowly increasing frequency is applied to an actuator with constant parameters. In both cases, the linear attachment is time-invariant, and the system is initially engaged in resonance. It is shown that in case (1) AR the nonlinear oscillator generates oscillations with growing amplitudes in the attached chain, while in case (2) energy transfer from the nonlinear oscillator is insufficient to excite high-energy motion in the attachment. The difference in the dynamical behavior is explained by different resonance properties of the systems. It is also shown that a slow change of stiffness may enhance the response of the nonlinear oscillator and make it sufficient to support oscillations with growing energy in the linear attachment even beyond the linear resonance. Explicit asymptotic approximations of the solutions are obtained.

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1. Introduction

Resonant energy transfer from a source of energy to a receiver has been identified as a one of the most effective methods for exciting and controlling high-energy oscillations in a broad range of physical and engineering systems. An idea of control intended to sustain “resonance under action of the force produced by the system’s itself” was suggested by Andronov¹. Feedback control schemes building on this idea and using self-sustained oscillations with predefined energy as a working process were implemented in a number of engineering systems². Although feedback

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control provides effective self-tuning and self-adaptation mechanisms targeted at preserving resonant oscillations under variations of structural parameters, its implementation requires careful diagnostics of nonlinear states and may become extremely complicated in a multi-dimensional system. A large class of systems can avoid feedback, still producing the required state with the help of properly chosen program control of resonant motion. This type of control employs an intrinsic property of a nonlinear oscillator to change both its amplitude and natural frequency when the driving frequency changes. This means that the oscillator may be persistently captured into resonance with its drive if the driving frequency varies slowly in time to be consistent with the slowly changing amplitude and frequency of the oscillator³⁻⁵. The ability of a nonlinear oscillator to stay captured into resonance due to variance of its structural or/and excitation parameters is known as *autoresonance* (AR).

Autoresonance was first used in applications to particle acceleration and reported as “the phase stability principle”^{6,7}. Building on that works, a large number of theoretical investigations, experimental results, and applications of AR in different fields of natural science have been reported in literature⁸⁻¹¹.

This paper investigates for the first time energy transfer and localization in a resonant multidimensional array consisting of a chain of time-invariant linear oscillators with equal partial frequencies coupled to a nonlinear actuator (the Duffing oscillator) driven by an external force. Two types of system are considered: (1) an actuator with slowly time-decreasing linear stiffness is driven by a periodic force with constant frequency; (2) a time-independent nonlinear actuator is driven by a force with a slowly-increasing frequency. In both cases, the parameters of the chain and coupling remain constant, and the system is initially captured into resonance. The purpose of this paper is to find the conditions under which AR in the nonlinear actuator brings about growing oscillations in the coupled chain. In Section 2 we show that, under certain conditions, AR in the nonlinear actuator gives rise to AR in the entire system of the first type. However, as shown in Sec. 3, in the system of the second type AR can occur only in the excited nonlinear actuator while the response of the coupled linear chain remains bounded. It is proved that additional slow variations of the linear stiffness enhance the response of the nonlinear actuator and makes it sufficient to excite AR in the coupled chain. It is shown that the difference in the dynamical behavior arises due to different resonance properties of the systems.

2. AR in a system subjected to a periodic force with a constant frequency

In this section we investigate an oscillator array consisting of a chain of n linear oscillators coupled to a nonlinear actuator (the Duffing oscillator). An external periodic force is applied solely to the actuator. Assuming a linear coupling between the entities, uniaxial motion of the array is governed by the following equations:

$$\begin{aligned} m_0 \frac{d^2 u_0}{dt^2} + [c_0 - (\kappa_1 + \kappa_2 t)] u_0 + \gamma u_0^3 + c_{0,1}(u_0 - u_1) &= A \cos \omega t, \\ m_k \frac{d^2 u_k}{dt^2} + c_k u_k + c_{k,k-1}(u_k - u_{k-1}) + c_{k,k+1}(u_k - u_{k+1}) &= 0, k = 1, \dots, n-1, \\ m_n \frac{d^2 u_n}{dt^2} + c_n u_n + c_{n,n-1}(u_n - u_{n-1}) &= 0. \end{aligned} \quad (1)$$

In (1), u_k represents the displacement of the k -th oscillator from the equilibrium, m_k is its mass; the coefficients c_k and $c_{k,k+1} = c_{k+1,k}$ are the linear stiffness constants of the k -th oscillator and stiffness of linear coupling between the k -th and $(k+1)$ -th oscillators, respectively; $\kappa_{1,2} > 0$ are constant detuning parameters; it will be shown that a proper choice of the parameters $\kappa_{1,2}$ may serve as a tool for controlling AR in the array. The parameters A and ω denote the amplitude and the frequency of the excitation. The system is initially at rest, i.e., $u_k = 0$, $du_k/dt = 0$ at $t = 0$.

Nonlinear system (1) can be analyzed asymptotically. The small parameter ε is introduced as a dimensionless coupling parameters by formula $\varepsilon = c_{1,0}/2c_1 \ll 1$. Taking into account resonance properties of the system and assuming weak nonlinearity, we redefine the system parameters as follows:

$$\begin{aligned} c_k / m_k &= \omega^2, \tau_0 = \omega t, \tau = \varepsilon s \tau_0, A = 2\varepsilon m_0 \omega^2 F, \\ \kappa_1 / c_0 &= 2\varepsilon s, \kappa_2 / c_0 = 2\varepsilon^2 \beta s^2 \omega, \gamma / c_0 = 8\varepsilon \alpha, c_{k,k+1} / c_k = 2\varepsilon \lambda_{k,k+1}. \end{aligned} \quad (2)$$

In these notations, equations (1) can be rewritten as

$$\begin{aligned} \frac{d^2 u_0}{d\tau_0^2} + (1 - 2\varepsilon s \zeta_0(\tau)) u_0 + 2\varepsilon \lambda_0 (u_0 - u_1) + 8\varepsilon \alpha u_0^3 &= 2\varepsilon F \sin \tau_0, \\ \frac{d^2 u_k}{d\tau_0^2} + u_k + 2\varepsilon \lambda_{k,k-1} (u_k - u_{k-1}) + 2\varepsilon \lambda_{k,k+1} (u_k - u_{k+1}) &= 0, k = 1, \dots, n \end{aligned} \quad (3)$$

with initial conditions $u_k = 0$, $v_k = du_k/d\tau_0 = 0$ at $\tau_0 = 0$. By definition, $\zeta_0(\tau) = 1 + \beta\tau$, $\lambda_{0,1} = \lambda_0$, $\lambda_{1,0} = 1$, $\lambda_{k,k+1} \neq \lambda_{k+1,k}$ but $\lambda_{n+1,n} = \lambda_{n,n+1} = 0$.

System (3) can be investigated with the help of the multiple scales method¹². To this end, we introduce complex amplitudes Ψ_k and define new rescaled parameters by formulas

$$\Psi_k = A^{-1} (v_k + i u_k) e^{-i\tau_0}, A = (s/(3\alpha))^{1/2}, \quad (4)$$

$$f = F/(sA), \mu_{k,k+1} = \lambda_{k,k+1}/s, \mu_{0,1} = \mu_0, k = 0, 1, \dots, n.$$

In terms of new variables, Eqs. (3) can be converted into the system in the standard form with the right-hand side proportional to ε :

$$\begin{aligned} \frac{d\Psi_0}{d\tau_0} &= -i\varepsilon s [\zeta_0(\tau) - |\Psi_0|^2] \Psi_0 + i\varepsilon s [\mu_0(\Psi_0 - \Psi_1) - f + G_0], \Psi_0(0) = 0, \\ \frac{d\Psi_k}{d\tau_0} &= i\varepsilon s [\mu_{k,k-1}(\Psi_k - \Psi_{k-1}) + \mu_{k,k+1}(\Psi_k - \Psi_{k+1}) + G_k e^{-2i\tau_0}], \Psi_k(0) = 0, k = 1, \dots, n, \end{aligned} \quad (5)$$

where the terms G_0 and G_k include the sums of harmonics with the coefficients depending on the amplitudes Ψ_0 , Ψ_k and their complex conjugates Ψ_0^* , Ψ_k^* . Explicit expressions of G_0 and G_k are insignificant for further analysis.

It follows from (5) that the asymptotic representation of Ψ_k takes the form $\Psi_k(\tau_0, \tau, \varepsilon) = \psi_k(\tau) + \varepsilon \psi_k^{(1)}(\tau_0, \tau) + \dots$, $\tau = \varepsilon s \tau_0$. The slow terms $\psi_k(\tau)$ satisfy the following dimensionless equations:

$$\begin{aligned} \frac{d\psi_0}{d\tau} &= -i[\zeta_0(\tau) - |\psi_0|^2] \psi_0 + i\mu_0(\psi_0 - \psi_1) - if, \psi_0(0) = 0, \\ \frac{d\psi_k}{d\tau} &= i[\mu_{k,k-1}(\psi_k - \psi_{k-1}) + \mu_{k,k+1}(\psi_k - \psi_{k+1})], \psi_k(0) = 0, k = 1, \dots, n. \end{aligned} \quad (6)$$

The error estimate $|\Psi_k(\tau_0, \tau, \varepsilon) - \psi_k(\tau)| \rightarrow 0$ as $\varepsilon \rightarrow 0^{5,12}$. The real-valued amplitudes and phases of oscillations are defined as $a_r = |\psi_r|$, $\Delta_r = \arg \psi_r$.

2.1. Quasi-steady states and fast fluctuations

In the first step, we study in detail a weakly coupled system with a negligibly small effect of the attachment on the dynamics of the excited nonlinear oscillator. This implies that in the leading-order approximation the emergence of AR in both parts of the array can be studied separately.

We recall the conditions of the emergence of AR in a single Duffing oscillator. It was shown^{13,14} that AR in the Duffing oscillator may occur if $f > f_1 = \sqrt{2/27} \approx 0.272$; if $f < f_1$ oscillator exhibits bounded oscillations at any detuning rate β . On the other hand, in the domain $f > f_1$ the Duffing oscillator admits AR if $\beta < \beta^*$; if $\beta > \beta^*$ the oscillator exhibits bounded oscillations (the regime of saturation). The critical detuning rate was defined as $\beta^* = [(f/f_1)^{2/3} - 1]/T^*$, where $\tau = T^*$ corresponds to the first minimum of the phase $\Delta_0(\tau)$ in the time-independent Duffing oscillator ($\beta = 0$). The value T^* was also found both numerically and analytically. We choose the parameters $f > f_1$, $\beta < \beta^*$ to ensure the existence of AR in a single Duffing oscillator. It is important to note that these inequalities cannot be considered as rigorous conditions of the occurrence of AR in the nonlinear oscillator included in the chain but they indicate admissible parametric intervals, in which AR can exist.

As in a single oscillator, the complex envelope $\psi_0(\tau)$ of the nonlinear actuator in the chain can be represented as a superposition of relatively small fast fluctuations $\tilde{\psi}_0(\tau)$ near the quasi-steady state $\bar{\psi}_0(\tau)$, i.e., $\psi_0(\tau) = \bar{\psi}_0(\tau) + \tilde{\psi}_0(\tau)$, where the state $\bar{\psi}_0$ is calculated as a stationary point of system (6) with the “frozen” parameter ζ_0 . Assuming $\mu_{k,k+1} = o(1)$, $f = o(1)$, we obtain

$$\bar{\psi}_0 \approx \pm \sqrt{\zeta_0}, \bar{a}_0 = |\bar{\psi}_0| \approx \sqrt{\zeta_0} \rightarrow \sqrt{\beta\tau}, \tau \rightarrow \infty \quad (7)$$

with the values of the phases $\bar{\Delta}_0 = 0$ or $\bar{\Delta}_0 = \pi$. Once the solution $\bar{\psi}_0(\tau)$ is known, then asymptotic approximations for fluctuations $\tilde{\psi}_0(\tau)$ can be found from the equation linearized near the state $\bar{\psi}_0$ ¹⁵.

In order to calculate the response of the chain, we rewrite the linear subsystem of (6) in the vector form:

$$\frac{d\Psi}{d\tau} + iM\Psi = -i\mu_{1,0}R\psi_0(\tau), \Psi(0) = 0, \quad (8)$$

where $\Psi = (\psi_1, \dots, \psi_n)^T$, $R = (1, 0, \dots, 0)^T$, M is the matrix of the coefficients. It follows from (8) that

$$\Psi(\tau) = -i\mu_{1,0}e^{-iM\tau} \int_0^\tau e^{iMs} R[\bar{\psi}_0(s) + \tilde{\psi}_0(s)] ds. \quad (9)$$

Taking into account a negligible effect of small fast fluctuations $\tilde{\psi}_0(\tau)$ on the value of integral (9), one obtains

$$\Psi(\tau) \approx -i\mu_{1,0}e^{-iM\tau} J(\tau), J(\tau) = \int_0^\tau e^{iMs} \bar{\psi}_0(s) ds R. \quad (10)$$

Integration by parts gives

$$J(\tau) = -iM^{-1}[e^{iM\tau}\bar{\psi}_0(\tau) - I\bar{\psi}_0(0)]R - \Phi(\tau),$$

$$\Phi(\tau) = \int_0^\tau e^{iMs} (d\bar{\psi}_0/ds) ds R,$$

where $\bar{\psi}_0(\tau) = \sqrt{1 + \beta\tau}$, $d\bar{\psi}_0/ds = \beta/[2\sqrt{1 + \beta s}]$. Each component ϕ_k of the vector Φ can be expressed as $\phi_k(\tau) = \sqrt{\beta}\Sigma_k(\tau)$, where $\Sigma_k(\tau)$ is a sum of the bounded Fresnel integrals. Hence $|\Sigma_k(\tau)| < C_k$, and

$$\Psi(\tau) = \bar{\Psi}(\tau) + \tilde{\Psi}(\tau) + O(\sqrt{\beta}), \bar{\Psi}(\tau) = -i\mu_{1,0}M^{-1}R\bar{\psi}_0(\tau), \tilde{\Psi}(\tau) = -e^{-iM\tau}\bar{\Psi}(0). \quad (11)$$

It now follows from (6), (11) that the response of the k -th oscillator can be expressed as $\psi_k(\tau) = \bar{\psi}_k(\tau) + \tilde{\psi}_k(\tau)$, where the quasi-steady states $\bar{\psi}_k(\tau)$ ($k = 1, \dots, n$) are given by the equality $\bar{\psi}_k(\tau) = \bar{\psi}_0(\tau)$. Hence, AR in the actuator gives rise to oscillations with growing amplitudes in the coupled chain.

As an example, we investigate the emergence of AR in the chain of 3 equal linear oscillators linearly coupled to the forced Duffing oscillator. In this case, system (6) is rewritten as

$$\begin{aligned} \frac{d\psi_0}{d\tau} + i[\zeta_0(\tau_1) - |\psi_0|^2]\psi_0 - i\mu_0(\psi_0 - \psi_1) &= -if, \\ \frac{d\psi_1}{d\tau} - i\mu(2\psi_1 - \psi_2) &= -i\mu\psi_0, \\ \frac{d\psi_2}{d\tau} - i\mu(2\psi_2 - \psi_1 - \psi_3) &= 0, \\ \frac{d\psi_3}{d\tau} - i\mu(\psi_3 - \psi_2) &= 0. \end{aligned} \quad (12)$$

It can be easily shown that the quasi-steady states $\bar{\psi}_k(\tau) = \bar{\psi}_0(\tau) = \sqrt{1 + \beta\tau}$, $k = 1, 2, 3$. Figure 1 depicts the amplitudes of oscillations $a_k = |\psi_k|$ in the system with parameters

$$\beta = 0.05, f = 0.34, \mu_0 = 0.015, \mu = 0.1 \quad (13)$$

and zero initial conditions. The characteristic polynomial of the linear subsystem is expressed as $D(s) = (s - i\mu)(s - 2i\mu)^2 + \mu^2(2s - 3i\mu)$. The roots of the characteristic equation $D(s) = 0$ are given by $s_r = i\omega_r$, where $\omega_r = \mu\alpha_r$, $\alpha_1 = 0.2$, $\alpha_2 = 1.56$, $\alpha_3 = 3.26$. The period of the dominant low-frequency harmonic $T_1 = 2\pi/\omega_1 = 314$ is obviously close to the exact (numerical) value $T \approx 350$ (Figs. 1(b) - 1(d)); the difference between the analytical and numerical results is about 10%. Initial conditions $\tilde{\psi}_k(0) = -1$ define the following amplitudes of the dominant low-frequency harmonics ω_1 in the k -th oscillator ($k = 1, 2, 3$): $a_{11} = 0.63$; $a_{21} = 0.97$, $a_{31} = 1.22$. It is easy to verify that, according to Fig. 1, the amplitudes a_{k2} and a_{k3} corresponding to higher-frequency harmonics satisfy the conditions $a_{k2} \ll a_{k1}$, $a_{k3} \ll a_{k1}$ ($k = 1, 2, 3$).

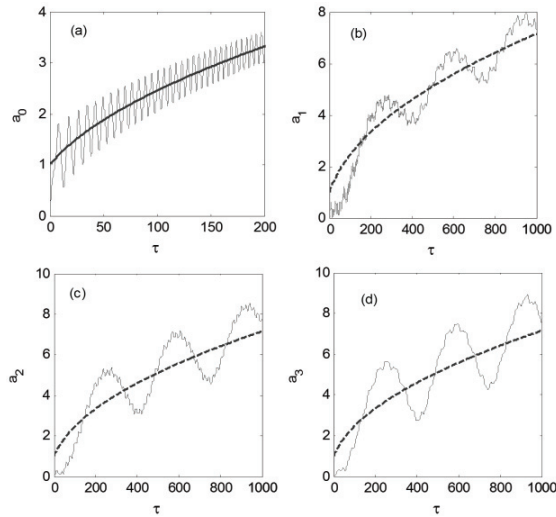


Fig. 1. Amplitudes of autoresonance oscillations; dashed lines correspond to the curves $\bar{a}_r = \bar{a}_0$.

3. Localization and transfer of energy in a system with a slowly increasing forcing frequency

In this section we study the behavior of the system of the second type, wherein the time-invariant nonlinear actuator is subjected to a periodic excitation with a frequency linearly increasing in time.

In the system with constant parameters the linear subsystem is identical to that one in (3) but the equation of the actuator takes the form

$$\begin{aligned} \frac{d^2 u_0}{d\tau_0^2} + u_0 + 2\varepsilon\lambda_0(u_0 - u_1) + 8\varepsilon\alpha u_0^3 &= 2\varepsilon F \sin(\tau_0 + \theta_0(\tau)), \\ \frac{d\theta_0}{d\tau} &= \zeta_0(\tau), \tau = \varepsilon\tau_0, \end{aligned} \quad (14)$$

with the coefficients defined by formulas (2). Transformations (4) - (6) together with the change of variables $\phi_k(\tau) = \psi_k(\tau)e^{-i\theta_0(\tau)}$ yield the following equations for the dimensionless complex amplitudes ϕ_k :

$$\begin{aligned} \frac{d\phi_0}{d\tau} + i[\zeta_0(\tau) - |\phi_0|^2]\phi_0 - i\mu_0(\phi_0 - \phi_1) &= -if, \\ \frac{d\phi_k}{d\tau} + i\zeta_0(\tau)\phi_k - i[\mu_{k,k-1}(\phi_k - \phi_{k-1}) + \mu_{k,k+1}(\phi_k - \phi_{k+1})] &= 0, \end{aligned} \quad (15)$$

with zero initial conditions. It is important to note that slow detuning $\zeta_0(\tau) = 1 + \beta\tau$ is now involved in all equations.

As in the previous section, the solution $\phi_0(\tau)$ is represented as the sum $\phi_0(\tau) = \bar{\phi}_0(\tau) + \tilde{\phi}_0(\tau)$, where $\bar{\phi}_0(\tau)$ and $\tilde{\phi}_0(\tau)$ denote the quasi-steady state of the nonlinear oscillator and fast fluctuations near $\phi_0(\tau)$, respectively. Under the conditions $\mu_0 \sim o(1)$, $f \sim o(1)$ the function $\phi_0(\tau)$ satisfies the equation similar to (7), so that $\bar{\phi}_0(\tau) \approx \sqrt{\zeta_0(\tau)}$.

After computing $\phi_0(\tau)$, all other variables can be found from (15). In analogy to (8), the linear part of (15) can be represented as

$$\frac{d\Phi}{d\tau} + iM_1(\tau)\Phi = -i\mu_{1,0}R\phi_0(\tau), \Phi(0) = 0. \quad (16)$$

We now show that each component of the vector $\Phi(\tau)$ is bounded, i.e. $|\phi_k(\tau)| < c_k < \infty$, $\tau \geq 0$. For brevity, the model of two coupled oscillators is considered. The slow dynamics of this system is described by the equations

$$\begin{aligned} \frac{d\phi_0}{d\tau} + i[\zeta_0(\tau) - |\phi_0|^2]\phi_0 - i\mu_0(\phi_0 - \phi_1) &= -if, \\ \frac{d\phi_1}{d\tau} + i[\zeta_0(\tau) - \mu]\phi_1 &= -i\mu\phi_0 \end{aligned} \quad (17)$$

with zero initial conditions. The change of variables $S(\tau) = (1 + \beta\tau)^2$ and simple transformations yield

$$\begin{aligned} \phi_1(\tau) &\approx -i\frac{\mu}{2\beta}e^{-iS(\tau)/(2\beta)}K(\tau), \\ K(\tau) &= K_0(\tau) - K_0(1), K_0(\tau) = \int_0^{S(\tau)} e^{iz/(2\beta)} z^{-1/4} dz. \end{aligned} \quad (18)$$

Although the function $K_0(\tau)$ cannot explicitly computed, the limiting value $K_0(\infty)$ can be calculated, and equals $K_0(\infty) = (2\beta)^{4/3}\Gamma(3/4)e^{3i\pi/8}$, where Γ is the gamma function. Hence $a_1(\tau) = |\phi_1(\tau)| \rightarrow \mu(2\beta)^{1/3}\Gamma(3/4)$ as $\tau \rightarrow \infty$. Therefore, the most part of the excitation energy remains localized on the excited oscillator but the rest of energy transferred to the linear oscillator suffices to sustain motion with a bounded but non-growing amplitude. Similar reasoning applied to the multi-dimensional array allows concluding that the response of each linear oscillator is bounded.

Numerical results have been obtained for the four-dimensional chain. The amplitudes of oscillations $a_k(\tau) = |\phi_k(\tau)|$ are shown in Fig. 2.

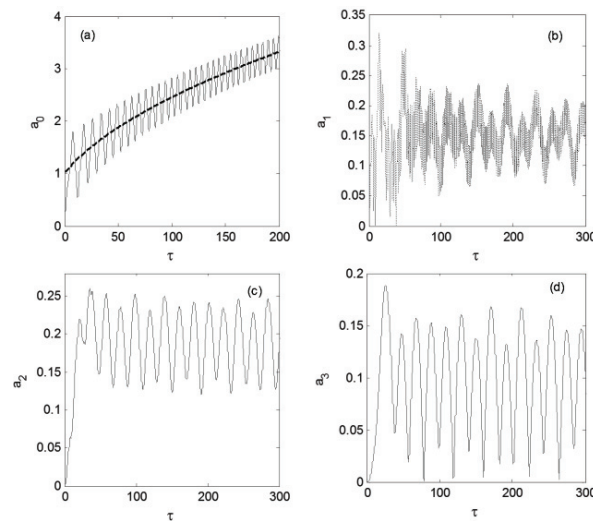


Fig. 2. Amplitudes of oscillations of the actuator and the coupled oscillators in the 4-dimensional chain.

The obtained numerical results depict a bounded response of all linear oscillators despite the presence of AR with a permanently growing mean amplitude in the forced nonlinear oscillator. This means that energy transfer from the excited oscillator is insufficient to render oscillations with increasing energy in the linear chain during escape from resonance.

It is important to note that escape from resonance does not directly prevent the growth of energy, as the linear chain is actually driven by the coupling response with the permanently increasing amplitude, and the system dynamics depends on the relationship between the growth of incoming energy and the loss of energy in the linear chain due to exit from resonance. We show that a slow change of linear stiffness of the actuator enhances its response and makes it sufficient to sustain growing oscillations in the coupled linear chain. Let the equation of the actuator be given by

$$\begin{aligned} \frac{d^2 u_0}{d\tau_0^2} + (1 - 2\varepsilon \xi_1(\tau))u_0 + 2\varepsilon \lambda_0(u_0 - u_1) + 8\varepsilon \alpha u_0^3 &= 2\varepsilon F \sin(\tau_0 + \theta_0(\tau)), \\ \frac{d\theta_0}{d\tau} &= \zeta_1(\tau), \end{aligned} \quad (19)$$

where $\zeta_1(\tau) = 1 + \beta_1 \tau$, $\xi_1(\tau) = \beta_3 \tau^3$. As in the previous example, transformations (4) - (6) and the change of variables $\psi_r(\tau) = \phi_r(\tau)e^{i\theta_0(\tau)}$ yield the following equations for the slow complex amplitudes $\phi_k(\tau)$:

$$\begin{aligned} \frac{d\phi_0}{d\tau} + i[\zeta_1(\tau) - |\phi_0|^2]\phi_0 - i\mu_0(\phi_0 - \phi_1) &= -if, \\ \frac{d\phi_k}{d\tau} + i\zeta_0(\tau)\phi_k - i[\mu_{k,k-1}(\phi_k - \phi_{k-1}) + \mu_{k,k+1}(\phi_k - \phi_{k+1})] &= 0, \end{aligned} \quad (20)$$

where $f = F/sA$, $\mu_r = \lambda_r/s$, $\mu_1 = s^{-1}$, $\mu_{n,n+1} = 0$, and $\zeta_0(\tau) = \zeta_1(\tau) + \xi_1(\tau)$. If $|\mu_{r,r\pm 1}| \sim o(1)$, $f \sim o(1)$, then the quasi-steady states and the corresponding amplitudes can be evaluated as

$$\bar{a}_0(\tau) = |\bar{\phi}_0(\tau)| \approx [\zeta_1(\tau) + \xi_1(\tau)]^{1/2}, \bar{a}_k(\tau) = |\bar{\phi}_k(\tau)| \approx \mu_{k,k-1} |\bar{\phi}_{k-1}(\tau)| / \zeta_1(\tau); \quad 1 \leq k \leq n. \quad (21)$$

As an example, we analyse the dynamics of the chain with 3 identical linear oscillators. We simulate numerically the system with modulations parameters $\zeta_1(\tau) = 1 + \beta_1 \tau$, $\xi_1(\tau) = \beta_3 \tau^3$; $\beta_1 = 10^{-2}$, $\beta_3 = 10^{-4}$. Forcing and coupling parameters are given by $f = 0.34$, $\mu_0 = 0.015$, $\mu = 0.1$. The results of numerical simulation in Fig. 3 depict increasing amplitudes of all linear oscillators.

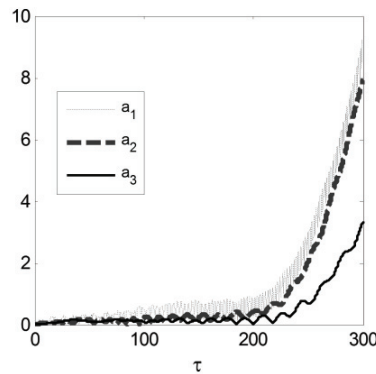


Fig. 3. Growing amplitudes of oscillations in the chain of 3 identical linear oscillators.

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References

1. Andronov AA, Vitt AA, Khaikin SE. *Theory of oscillators*. Oxford: Pergamon Press; 1966.
2. Astashev VK, Babitsky VI. *Ultrasonic processes and machines*. Berlin: Springer; 2007.
3. Neishtadt AI. Passage through a separatrix in a resonance problem with a slowly-varying parameter. *J Appl Math Mech* 1975; **39**: 594–605.
4. Neishtadt AI, Vasiliev AA, Artemyev AV. Capture into resonance and escape from it in a forced nonlinear pendulum. *Regul Chaotic Dyn* 2013; **18**: 691–701.
5. Arnold VI, Kozlov VV, Neishtadt AI. *Mathematical aspects of classical and celestial mechanics*. Berlin: Springer; 2006.
6. Veksler VI. A new method of accelerating relativistic particles. *Comptes Rendus (Doklady) de l'Academie Sciences de l'URSS* 1944; **43**: 329.
7. McMillan EM. The synchrotron – a proposed high-energy particle accelerator. *Phys Rev* 1945; **68**: 144.
8. Vazquez L, MacKay R, Zorzano MP. *Localization and energy transfer in nonlinear systems*. Singapore: World Scientific; 2003.
9. May V, Kühn O. *Charge and energy transfer dynamics in molecular systems*. Weinheim: Wiley – VCH; 2011.
10. Charman AE. *Random aspects of beam physics and laser-plasma interactions*. Berkley: University of California; 2007.
11. Chapman T. *Autoresonance in stimulated Raman scattering*. Paris: École Polytechnique; 2011.
12. Sanders JA, Verhulst F, Murdock J. *Averaging methods in nonlinear dynamical systems*. Berlin: Springer; 2007.
13. Kovaleva A, Manevitch LI. Limiting phase trajectories and emergence of autoresonance in nonlinear oscillators. *Phys Rev E* 2013; **88**: 024901.
14. Kovaleva A. Capture into resonance of coupled Duffing oscillators. *Phys Rev E* 2013; **92**: 022909.
15. Kovaleva A. Response enhancement in an oscillator chain. *Commun Nonlinear Sci Numer Simulat* 2016; **30**: 373–386.
<http://dx.doi.org/10.1016/j.cnsns.2015.07.003>.